ELSEVIER Journal of Geometry and Physics 24 (1998) 223–243

JOURNAL OF GEOMETRY AND PHYSICS

On the existence of superconformal 2-tori and doubly periodic affine Toda fields

Ian McIntosh [1]

School of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

Received 13 March 1996

Abstract

Recently Bolton, Pedit and Woodward (1995) described a class of harmonic surfaces (dubbed superconformal) in complex projective space corresponding to the affine Toda field equations for SU_{n+1}. This article gives a complete description of the conditions for the existence of superconformal 2-tori and doubly periodic Toda solutions in terms of differentials on the spectral curve. This shows that in $\mathbb{CP}^n$ for $n > 3$ one cannot expect to find superconformal 2-tori even though there should be plenty of doubly periodic Toda solutions.

Subj. Class.: Dynamical systems
1991 MSC: 58E20
Keywords: Toda fields; Superconformal 2-tori; Harmonic surfaces; Complex projective space; Symmetric space; Integrable systems

1. Introduction

Recently there has been a great deal of interest in the construction of harmonic surfaces in symmetric spaces associated to integrable systems (for example [4,5,8,12,16]). In particular, Bolton et al. [4] used the term superconformal to describe harmonic surfaces in $\mathbb{CP}^n$ associated to the 2D (affine) Toda field equations. By their definition a superconformal map is one whose harmonic sequence is orthogonally periodic. Directly from this one deduces that each superconformal *2-torus* determines a doubly periodic solution of the elliptic Toda equations

$$\frac{\partial^2}{\partial z \partial \bar{z}} \ln(s_j^2) = s_{j+1}^2 s_j^{-2} - s_j^2 s_{j-1}^{-2}, \tag{1}$$

[1] E-mail: i.r.mcintosh@maths.bath.ac.uk.

PII S0393-0440(97)00009-0

where z is a coordinate on the uniformising space $\mathbb{C}$ of the torus and each $s_0, \ldots, s_n$ is a strictly positive real-valued function of $z, \bar{z}$. Geometrically the 'fields' s_j arise from the differential $d\Psi$ of the harmonic map, which can be represented by a Lie algebra valued 1-form over the domain. Taking the domain to be a 2-torus enables us to have globally defined functions for the fields (otherwise we would have an equation in $(1, 1)$-forms). Conversely, given a doubly periodic solution to Eqs. (1) one integrates to find a Toda frame for a harmonic map $\mathbb{R}^2 \to \mathbb{CP}^n$: the obstruction to this map being doubly periodic is the monodromy of the frame around the periods of the Toda solution.

The main purpose of this note is to complete the study made in [15,16] by determining the conditions required for the existence of superconformal 2-tori. In particular, it is shown that these amount to a system of equations which appear to be over determined for $n \geq 2$ unless one considers tori with extra symmetry. This explains the 'dearth of examples' referred to in [5].

The strategy is to use the fact that every doubly periodic solution of Eqs. (1) belongs to the (strictly larger) class of solutions possessing a spectral curve in the sense of [15]. This class is usually referred to as the class of solutions of finite type. The aim is to distinguish inside this class first the doubly periodic Toda solutions and then all those whose Toda frame has trivial monodromy around both Toda periods. In Section 3 it is shown that this monodromy is measured by the scalar Baker function for the Toda solution. Consequently we are able (in Sections 4 and 5) to give expression to the periodicity conditions purely in terms of differentials on the curve.

From this we can make a naive parameter count for the number of free parameters available to satisfy these conditions. This count shows that while there are always enough free parameters to expect doubly periodic Toda solutions (indeed $2p - 1$ free parameters to satisfy $2p - 4$ conditions when the spectral curve has genus p) there are in general not enough free parameters to obtain superconformal tori except into $\mathbb{CP}^1$ (which are actually the non-conformal maps, see [2,3,14]) and $\mathbb{CP}^2$. Maps which have an S^1 symmetry have slightly fewer conditions and may also provide tori in $\mathbb{CP}^3$.

The final two sections deal with the important point of how one determines what the spectral curve of a Toda solution is. From the point of view of [15] the solution space given by dressing is a union of orbits of the infinite-dimensional abelian group of higher flows – it is shown that the solutions of finite type are precisely those with finite-dimensional orbit for this group *and* that this orbit is essentially the Jacobian of the spectral curve. Appendix A relates this to another point of view, expressed in [7,12], which constructs a spectral curve using 'polynomial Killing fields'. It also explains that the algebra of polynomial Killing fields is essentially the coordinate ring of the spectral curve used above, thus relating the two.

2. Superconformal maps and the Toda equations

This section summarises the link between superconformal tori and the elliptic Toda equations (following [4,6]), and explains why every doubly periodic solution arises from the dressing construction. To begin we recall the notion of a *primitive* harmonic map into

G/T (which is essentially the same as the term 'τ-primitive map' used in [4]) where T is the maximal torus of diagonal matrices in $G = SU_{n+1}$.

The full flag manifold G/T is an $n+1$-symmetric space, i.e. T is the fixed point subgroup of a periodic automorphism ν on G which has period $n+1$. This automorphism, called the Coxeter automorphism, is defined by $\nu(g) = \sigma g \sigma^{-1}$ where σ^{-1} is the diagonal matrix $\mathrm{diag}(1, \omega, \ldots, \omega^n)$ and ω is the $n+1$-th root of unity $\exp(2\pi i/(n+1))$. This induces an automorphism on the Lie algebra $\mathfrak{g} = \mathfrak{su}_{n+1}$ which is completely characterised by its eigenspace decomposition of the complexification $\mathfrak{g}^{\mathbb{C}}$: we write this as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \mathfrak{g}_1 + \cdots + \mathfrak{g}_n.$$

Here $\mathfrak{t}^{\mathbb{C}}$ is the complexification of $\mathfrak{t}$, the Lie algebra of T. The subspaces $\mathfrak{g}_k$ are the eigenspaces for eigenvalue ω^k. In particular $\mathfrak{g}_1$ consists of all the matrices in $\mathfrak{sl}_{n+1}$ of the form

$$\begin{pmatrix} 0 & * & & & \\ & \cdot & * & & \\ & & \cdot & \cdot & \\ & & & \cdot & * \\ * & & & & 0 \end{pmatrix},$$

where $*$ denotes a possible non-zero entry. An element of $\mathfrak{g}_1$ is called *cyclic* if all these entries are non-zero and it is readily seen that every cyclic matrix is, up to scaling, in the $Ad\ T^{\mathbb{C}}$-orbit of the cyclic matrix Λ which has 1's in every non-zero entry.

Let ξ^* denote the complex conjugate transpose of an element in $\mathfrak{g}^{\mathbb{C}}$. The conjugation $\xi \mapsto -\xi^*$ commutes with ν and the splitting $\mathfrak{g} = \mathfrak{t} + \mathfrak{m}$ is reductive when $\mathfrak{m}^{\mathbb{C}}$ denotes $\mathfrak{g}_1 + \cdots + \mathfrak{g}_n$. Now recall (from e.g. [10]) that the Maurer–Cartan form β for G/T is the endomorphism of vector bundles which identifies the tangent bundle $T(G/T)$ with the bundle $[\mathfrak{m}] = G \times_T \mathfrak{m}$, where T acts adjointly on $\mathfrak{m}$.

Throughout this article $\mathbb{T}^2$ will denote a 2-torus viewed as the quotient of the complex plane $\mathbb{C}$ by a lattice: since harmonic maps from a Riemann surface depend only on the conformal structure induced by the metric we need not be more precise. We will say a real-analytic map $\phi : \mathbb{T}^2 \to G/T$ is *primitive* if $\phi^*\beta^{(1,0)}$ takes values in $[\mathfrak{g}_1] = G \times_T \mathfrak{g}_1$. One knows from [1] that all primitive maps are equi-harmonic (i.e. harmonic with respect to any G-invariant metric on G/T) provided the order of ν is greater than 2 (i.e. $n > 1$).

Now let $\pi_k : G/T \to \mathbb{CP}^n$ be one of the $n+1$ possible homogeneous projections. Then one also knows from [1] that $\pi_k \circ \phi$ is harmonic whenever ϕ is primitive. This allows Bolton et al. [4] to characterise the superconformal tori in $\mathbb{CP}^n$ as follows. *A harmonic torus $\phi_k : \mathbb{T}^2 \to \mathbb{CP}^n$ is superconformal if and only if $\phi_k = \pi_k \circ \phi$ for some primitive $\phi : \mathbb{T}^2 \to G/T$ for which $\phi^*\beta^{(1,0)}$ is cyclic.*

Recall from e.g. [4] that given smooth solutions s_j to (1) (say, throughout $\mathbb{C}$) there is, for each $\zeta \in \mathbb{C}^*$, a unique $G^{\mathbb{C}}$-valued solution Φ_ζ to

$$\Phi_\zeta^{-1}\partial\Phi_\zeta = -s^{-1}\partial s - \zeta s^{-1}\Lambda s, \qquad \Phi_\zeta^{-1}\bar{\partial}\Phi_\zeta = s^{-1}\bar{\partial}s + \zeta^{-1}s\Lambda^* s^{-1}. \qquad (2)$$

normalised by $\Phi_\zeta(0) = Id$, where $\partial = \partial/\partial z$ and $s = \mathrm{diag}(s_0, \ldots, s_n)$. Moreover Φ_ζ is G-valued on $|\zeta| = 1$. The form of (2) is such that, for each ζ of unit modulus, $\Phi_\zeta^{-1}\partial\Phi_\zeta$ takes values in $\mathfrak{t}^{\mathbb{C}} + \mathfrak{g}_1$ with cyclic $\mathfrak{g}_1$-component. It follows that the map $\phi_\zeta : \mathbb{C} \to G/T$ defined by $\phi_\zeta = \Phi_\zeta T$ is primitive and each $\pi_k \circ \phi_\zeta$ gives a superconformal plane in $\mathbb{CP}^n$. The frame $\Phi = \Phi_1$ is called a Toda frame for $\phi = \phi_1$ and Φ_ζ is called an extended Toda frame.

Theorem 1 [4]. *Let $\phi : \mathbb{T}^2 \to G/T$ be primitive with cyclic $\phi^*\beta^{(1,0)}$, then $\phi = \Phi T$ for some doubly periodic $\Phi : \mathbb{C} \to G$ satisfying* (2) *with positive definite s. The functions s_j are doubly periodic with the same periods as $\mathbb{T}^2$ but Φ may change by a factor $\omega^k I$ around these periods. Moreover, Φ is uniquely determined by ϕ and the coordinate z up to translation of z or rescaling z by some power of ω.*

Moreover it is shown in [4] that, up to scaling, distinct solutions of (1) lead to distinct primitive maps ϕ. The next aim is to describe how every doubly periodic solution of Eqs. (1) arises from the dressing construction.

2.1. *The dressing construction*

For each value of $z, \bar{z}$ we can think of Φ_ζ as an element of a loop group of maps $S^1 \to G$. Indeed we may always choose Φ_ζ to be *ν-equivariant*, that is, $\nu(\Phi_\zeta) = \Phi_{\omega\zeta}$, since the right-hand side of (2) has this property in the loop algebra. The dressing construction used in [9] requires that we introduce the following loop groups.

Let $C = C_1 \cup C_2$ be the union of two circles in the ζ-plane, centred at $\zeta = 0$ and with radii ϵ and ϵ^{-1}, respectively (where $0 < \epsilon < 1$). We consider C to be the boundary of two regions on the Riemann sphere: the union of discs I and the annulus E. Define $\Lambda_C(G^{\mathbb{C}}, \nu)$ to be the group of real-analytic ν-equivariant maps $C \to G^{\mathbb{C}}$ (we may think of such a map as a pair (g_1, g_2) of ν-equivariant loops in $G^{\mathbb{C}}$). We will be primarily interested in the subgroup

$$\Lambda(G, \nu) = \{g \in \Lambda_C(G^{\mathbb{C}}, \nu) \colon\ g = \bar{g}\},$$

where $\bar{g}$ denotes $g(\bar{\zeta}^{-1})^{*-1}$. It is a corollary of a result in [16] that $\Lambda(G, \nu)$ has a global decomposition (which we will call its Iwasawa decomposition) into the product of the following three subgroups:

$$\begin{aligned}
\Lambda_E &= \{g \in \Lambda(G, \nu)\colon\ \text{boundaries of holomorphic maps } g : E \to G^{\mathbb{C}}\};\\
\Lambda_I &= \{g \in \Lambda(G, \nu)\colon\ \text{boundaries of holomorphic maps } g : I \to G^{\mathbb{C}}\\
&\qquad\qquad \text{with } g(0) = Id\};\\
D &= \{(s, s^{-1})\colon\ s \in \exp(i\mathfrak{t}) \text{ i.e. } s \text{ is diagonal, constant and positive definite}\}.
\end{aligned}$$

The decomposition asserts that every $g \in \Lambda(G, \nu)$ has a unique factorisation $g = udn$ where $u \in \Lambda_E$, $d \in D$ and $n \in \Lambda_I$.

The dressing construction uses this factorisation to 'dress up' the frame $\Phi_\zeta^{(0)} = \exp(-z\zeta\Lambda + \bar{z}\zeta^{-1}\Lambda^*)$ which corresponds to the vacuum solution of (1), where every $s_j = 1$.

Theorem 2 [16]. *Let $g \in \Lambda(G, \nu)$ and consider $\Phi_\zeta^{(0)}$ as a $\Lambda(G, \nu)$-valued function of $z, \bar{z}$. Then $g\Phi_\zeta^{(0)} = \Phi_\zeta dn$ according to the Iwasawa decomposition where Φ_ζ satisfies* (2) *for some possibly non-trivial s given by $d = (s, s^{-1})$. The entries s_j of the diagonal matrix s satisfy* (1).

Indeed whenever Φ_ζ satisfies (2) and $g \in \Lambda_C(G^C, \nu)$ the Λ_E-factor of $g\Phi_\zeta$ (which shall be denoted by $g\sharp\Phi_\zeta$) also satisfies (2) for a possibly different Toda solution. The set of such extended frames forms an orbit of the group $\Lambda_{I,D}$ generated by D and Λ_I. In fact:

Theorem 3. *Every doubly periodic solution of* (1) *possesses an extended Toda frame of the form $g\sharp\Phi^{(0)}$ for some $0 < \epsilon < 1$ and some $g \in \Lambda_{I,D}$.*

This is a consequence of results from [4,8,9] (compare [17]). It will be useful to briefly sketch the proof.

First we require some definitions and notation. The Lie algebra $\Lambda(\mathfrak{g}, \nu)$ of $\Lambda(G, \nu)$ splits into the direct (vector space) sum of Lie subalgebras

$$\Lambda(\mathfrak{g}, \nu) = \Lambda_E(\mathfrak{g}, \nu) + \mathfrak{d} + \Lambda_I(\mathfrak{g}, \nu)$$

corresponding to the Iwasawa decomposition of $\Lambda(G, \nu)$. For any positive integer k let $\Lambda_k \subset \Lambda_E(\mathfrak{g}, \nu)$ denote the subspace consisting of Laurent polynomials of degree $\leq k$. Now define the vector subspace

$$\Delta = \Lambda_1 + \mathfrak{d} + \Lambda_I(\mathfrak{g}, \nu) \subset \Lambda(\mathfrak{g}, \nu)$$

and, for any $(\bar{\xi}, \xi)$ in Δ, define $e(\xi) = \exp[(-\bar{z}\bar{\xi}, -z\xi)]$. This presents us with a 2-parameter subgroup of $\Lambda(G, \nu)$. According to the Iwasawa decomposition we may write

$$e(\xi) = \Phi(\xi)d(\xi)n(\xi). \tag{3}$$

Observe that $\Phi(\xi)$ is trivial unless ξ has a non-trivial component in Λ_1. For *any* ξ in Δ, $\Phi(\xi)$ satisfies equations very much like (2). Let us write the Fourier series for ξ (which lives on C_2) as $\xi = \xi_1\zeta + \xi_0 + \cdots$ then one readily shows that $\Phi(\xi)$ satisfies

$$\Phi^{-1}\partial\Phi = -s^{-1}\partial s - \zeta s^{-1}\xi_1 s, \qquad \Phi^{-1}\bar{\partial}\Phi = s^{-1}\bar{\partial}s - \zeta^{-1}s\bar{\xi}_1 s^{-1}, \tag{4}$$

where s comes from the factor $d(\xi) = (s, s^{-1})$ in (3). When ξ_1 is cyclic these equations are equivalent to (2), but if ξ_1 is not cyclic their compatibility equations are a variation on the finite lattice Toda equations (see for example [16]).

Now, to prove Theorem 3 we recall from [4] that every doubly periodic Toda solution is of *finite type*. This means it has an extended Toda frame Φ_ζ which, for some $(\bar{\eta}, \eta) : \mathbb{R}^2 \to \Lambda_m$ with $m \equiv 1 \bmod (n+1)$, satisfies

$$d\eta = [\eta, \Phi_\zeta^{-1}d\Phi_\zeta] \quad \text{with } -\mathrm{i}\eta = \zeta^m s^{-1}\Lambda s + \zeta^{m-1}\partial \ln s + \cdots \tag{5}$$

When this occurs we can readily verify that, for $\xi = -\mathrm{i}\zeta^{1-m}\eta|_{z=0}$, $\Phi(\xi)$ satisfies (4) with $\xi_1 = s^{-1}(0)\Lambda s(0)$. Since this is exp(it)-conjugate to Λ we deduce $\Phi(\xi) = g\sharp\Phi(\zeta\Lambda)$, for some ϵ, by applying [9, Theorem 3.7]. I leave it to the reader to show that $\Phi(\zeta\Lambda) = \Phi^{(0)}$ to complete the proof of Theorem 3.

3. Baker functions

Given a doubly periodic solution s to (1) it need not be that the (normalised) extended frame Φ_ζ be periodic for any value of ζ. To understand its monodromy around periods of s we can use the matrix and scalar Baker functions from [15].

Set $\tilde{\Psi} = \Phi_\zeta s^{-1}$ and define $q = 2\partial \ln s$, then it is a simple matter to verify that

$$\tilde{\Psi}^{-1}\partial\tilde{\Psi} = -q - \zeta\Lambda. \tag{6}$$

For any matrix s of solutions to (1) there is always a formal series solution of (6) with Fourier series in ζ of the form

$$\Psi(z, \bar{z}; \zeta) = \exp(-z\zeta\Lambda)(Id + \mathrm{O}(\zeta^{-1})) \tag{7}$$

We will call Ψ a *matrix Baker function* for q whenever it satisfies (6) and has Fourier series (7) convergent for $|\zeta| \geq \epsilon^{-1}$. When it exists there is more than one matrix Baker function for each q. Let K_2 denote the group of all $\gamma \in C^\omega(C_2, G^{\mathrm{C}})$ which extend holomorphically to $|\zeta^{-1}| \leq \epsilon$, with $\gamma_2(\infty) = Id$, such that $\gamma_2\Lambda\gamma_2^{-1} = \Lambda$.

Lemma 1. *When s comes from the dressing construction there is an analytic $g_2 : C_2 \to G^{\mathrm{C}}$ for which $\Psi = g_2^{-1}\Phi_\zeta s^{-1}$ is a matrix Baker function. Every other matrix Baker function for q is of the form $\gamma_2\Psi$ where $\gamma_2 \in K_2$.*

Proof. Since we have assumed $g\Phi_\zeta^{(0)} = \Phi_\zeta dn$ for some g in $\Lambda(G, \nu)$ we have, on C_2, $g_2\Phi_\zeta^{(0)} = \Phi_\zeta s^{-1}n_2$ where $n_2 = n|C_2$. Therefore $g_2^{-1}\Phi_\zeta s^{-1} = \Phi_\zeta^{(0)} n_2^{-1}$ has Fourier series of the form (7) and satisfies (6). Moreover, it is convergent for $|\zeta| \geq \epsilon^{-1}$ since n_2 is. Clearly any other solution of (6) for the same q equals $\gamma_2\Psi$ for some $\gamma_2 : C_2 \to G^{\mathrm{C}}$. To satisfy (7) $\gamma(z) = \exp(z\zeta\Lambda)\gamma_2\exp(-z\zeta\Lambda)$ must be of the form $Id + \mathrm{O}(\zeta^{-1})$ for all z. In that case $(\partial^k\gamma)_{z=0} = \zeta^k ad^k\Lambda \cdot \gamma_2$ may only have terms in negative powers of ζ. Since Λ is semisimple it follows that γ_2 has no terms of any order which do not commute with Λ, thus $\gamma_2 \in K_2$. □

We will require a result from [15] concerning a space of solutions to the *complexified* Toda equations, which are obtained from (1) by replacing $-\bar{z}$ with a complex variable t independent of z. Let $\Lambda_E(G^{\mathrm{C}}, \nu)$ be the subgroup of $\Lambda_C(G^{\mathrm{C}}, \nu)$ consisting of boundaries of holomorphic maps $E \to G^{\mathrm{C}}$ and let $\mathcal{M}$ denote the left quotient space $\Lambda_E(G^{\mathrm{C}}, \nu) \setminus \Lambda_C(G^{\mathrm{C}}, \nu)$. Now let $K \subset \Lambda_C(G^{\mathrm{C}}, \nu)$ be the subgroup of boundaries of holomorphic maps $\gamma : I \to G^{\mathrm{C}}$ with $\gamma(\infty) = Id$ and $\gamma\Lambda\gamma^{-1} = \Lambda$ (this group is abelian since Λ has abelian stabilizer in G^{C}). K has identity component K_0. Every element of K is of the form $(\omega^k, 1)\gamma$ for some $\gamma \in K_0$ and some integer k. Hence K/K_0 is isomorphic to $\mathbb{Z}_{n+1}$. Given all this, the following theorem sums up (albeit in slightly different notation) the dressing construction described in [15].

Theorem 4. *The abelian group K acts freely on the left of $\mathcal{M}$ and the dressing construction produces the injective horizontal maps in the commutative diagram:*

$$\begin{array}{ccl} \mathcal{M}/K_0 & \longrightarrow & \{s^2 : s_j^2 \text{ satisfy the complexified Toda equations}\} \\ \downarrow & & \downarrow \\ \mathcal{M}/K & \longrightarrow & \{q = \partial \ln(s^2)\}. \end{array} \tag{8}$$

Each fibre of the right-hand vertical map, over a point in the image of the bottom horizontal map, is of the form $\{s^2, \omega s^2, \ldots, \omega^n s^2\}$. Moreover, if the double coset for some $g \in \Lambda_C(G^C, \nu)$ is mapped, under the top map, to $s^2(z,t)$ then the double coset for $g\exp(-z_0\zeta\Lambda - t_0\zeta^{-1}\Lambda^*)$ is mapped to $s^2(z+z_0, t+t_0)$.

This last fact is very important: it describes the flow governed by the Toda equations on the space $\mathcal{M}/K_0$, which we think of as a phase space containing a certain class of solutions. Each point in the space corresponds to an initial condition, while the z and t flows are generated simply by the right action of the 2-parameter abelian group $\{\exp(-z\zeta\Lambda - t\zeta^{-1}\Lambda^*) : z, t \in \mathbb{C}\}$ (which commutes with K). More generally, let Γ be the stabilizer of Λ in $\Lambda_E(G^C, \nu)$, then its natural action (on the right) on $\mathcal{M}$ covers an action on $\mathcal{M}/K_0$, since Γ commutes with K. This generates the 'higher commuting flows' in the Toda hierarchy. We will have more to say about these Γ-orbits later (Section 6).

Note. Beware that in [15] the loop algebra is identified with *right* invariant vector fields on the loop group, hence the quotient spaces there are the reverse of those here. Later on it will be useful to recognise that we can replace SL_{n+1} by GL_{n+1} in the description of $\mathcal{M}/K$ and Γ without changing the theorem (provided we work with loops of winding number zero).

3.1. The monodromy generators

Now we will use the results just stated to compute the eigenvalues of the monodromy matrices for a doubly periodic solution of (1).

Lemma 2. *The diagonal matrix s containing the solutions s_j for the elliptic Toda lattice (1) is doubly periodic if and only if q is. Moreover, they have the same periods.*

Proof. Clearly q is doubly periodic whenever s is. Conversely, by Theorem 4 we know that if q is doubly periodic then each s_j^2 changes by a multiplier which is some power of ω around each period. However, each s_j is a real-valued, strictly positive function, hence the multiplier of s_j around each period is 1. □

When q is doubly periodic, with periods ξ_1 and ξ_2, the matrix Baker function will have monodromy:

$$\Psi(z+\xi_j;\zeta) = M_j(\zeta)\Psi(z;\zeta). \tag{9}$$

By lemma 2, Φ_ζ will also have monodromy:

$$\Phi_\zeta(z+\xi_j) = m_j(\zeta)\Phi_\zeta. \tag{10}$$

By assumption Φ_ζ comes from the dressing construction, so $m_j(\zeta)$ is defined on the annulus E. The link between the two is given by

$$g_2 M_j g_2^{-1} = m_j, \tag{11}$$

which follows directly from Lemma 1. Notice that (11) will only be defined near C_2.

The necessary and sufficient condition for double periodicity of Φ_1 is $m_j(1)^r = Id$ for some positive integer r. To investigate when this occurs it is sufficient to study the eigenvalues of $m_j(\zeta)$. Since the eigenvalues are the solutions of the characteristic equation $\det(m_j(\zeta) - \mu \cdot Id) = 0$ they should be thought of as functions on a Riemann surface which covers the annulus of definition of $m_j(\zeta)$. Thus $m_j(1)^r = Id$ only if each of these eigenvalues is an rth root of unity at every point lying over $\zeta = 1$. With the help of (11) we can give a clear description of this condition, by first examining the eigenvalues of $M_j(\zeta)$.

Proposition 1. *For $|\zeta| \geq \epsilon^{-1}$ we have*

$$M_j(\zeta) = \exp\left(-\xi_j\zeta\Lambda + \sum_1^\infty a_j\zeta^{-j}\Lambda^{-j}\right) \tag{12}$$

for some sequence $\{a_j\}$. Therefore the k-th eigenvalue of $M_j(\zeta)$ is $\mu_j(\omega^k\zeta)$ where $\mu_j(\zeta) = \exp(-\xi_j\zeta + \sum_1^\infty a_j\zeta^{-j})$.

Proof. When q has periods ξ_j we see that $\exp(-\xi_j\zeta\Lambda)\Psi(z+\xi_j)$ is also a matrix Baker function for q, hence (by Lemma 1) it equals $\gamma_2\Psi$ for some $\gamma_2 \in K_2$. It is easy to verify that all elements of K_2 are exponentials of power series in negative powers of $\zeta\Lambda$, hence M_j is of the form (12). The last statement follows at once from the fact that Λ may be diagonalised into the matrix $\mathrm{diag}(1,\omega,\ldots,\omega^n)$. □

Now we will see that $\mu_j(\zeta)$ can be computed using the scalar Baker function defined as follows. Let $\psi_0^{\mathrm{row}}(z,\bar{z};\zeta)$ denote the top row of the matrix Ψ^{-1}. The ν-equivariance of Ψ means that

$$\psi_0^{\mathrm{row}}(\zeta) = (\psi_{00}(\zeta^{n+1}), \zeta\psi_{01}(\zeta^{n+1}), \ldots, \zeta^n\psi_{0n}(\zeta^{n+1})).$$

We may define the scalar Baker function $\psi(z,\bar{z};\zeta)$ to be the sum of the entries of ψ_0^{row} (that this coincides with the usual definition in [11,15,16,21] is explained in Section 4). It is easily seen that it has Fourier series in ζ of the form

$$\psi(\zeta) = \exp(z\zeta)(1 + \mathrm{O}(\zeta^{-1})).$$

Moreover, since Λ is a permutation matrix, summing together the entries of the row vector $\psi_0^{\mathrm{row}} \cdot (\zeta\Lambda)^{-j}$ gives $\zeta^{-j}\psi$. It follows from (10) and Proposition 1 that

$$\psi(z+\xi_j;\zeta) = \psi(z;\zeta)\mu_j(\zeta)^{-1}.$$

We immediately deduce that $\mu_j(\zeta) = \psi(0;\zeta)\psi(\xi_j;\zeta)^{-1}$. However, we may always choose ψ so that $\psi(0;\zeta) = 1$ (see Remark below). Using this normalisation (which is the standard one in e.g. [21]) we have

$$\mu_j(\zeta) = \psi(\xi_j;\zeta)^{-1}. \tag{13}$$

Remarks. That we can normalise ψ in this way follows from Lemma 1 since this implies that ψ is unique up to factors of the form $1 + \mathrm{O}(\zeta^{-1})$. The Fourier series for $\psi(0;\zeta)$ is of this form hence ψ can be normalised to 1 at $z = 0$.

4. The spectral curve

In the final section I will show that when q is doubly periodic it arises from the algebro-geometric construction given in [15,16] which sits inside the dressing construction of Theorem 4. In this section we recall the relevant parts of this construction. The aim here is to describe the link between the pair (q, ψ) and a complete algebraic curve X together with some other data.

First we recall from [16] that given $n+1$ scalar functions $\psi_j(z,t;\zeta)$ satisfying a system of the type

$$\partial_z\psi_j = q_j\psi_j + \zeta\psi_{j+1}, \quad \partial_t\psi_j = \zeta^{-1}v_j\psi_{j-1} \tag{14}$$

for some functions $q_j(z,t)$ and $v_j(z,t)$, then the compatibility conditions for these equations imply the complexified Toda lattice in the form

$$\partial_z\partial_t \ln(v_j) = -v_{j-1} + 2v_j - v_{j+1} \tag{15}$$

(where in both (14) and (15) j is counted modulo $n+1$). To pass from this form of Toda to the earlier form (1), set $t = -\bar{z}$) and $v_j = s_j^2 s_{j-1}^{-2}$.

We can obtain a set of functions ψ_j satisfying (14) for some q_j, v_j after fixing a Riemann surface X (or, more generally, a complete irreducible algebraic curve) together with a non-special line bundle $\mathcal{L}$ over X whose degree equals the arithmetic genus p of X. The curve X must admit a rational function $\pi : X \to \mathbb{CP}^1$ of degree $n+1$ with divisor of the form $(n+1)(P_0 - P_\infty)$ for two non-singular points P_0 and P_∞ on X. Using π we define local parameters ζ about P_0 and ζ^{-1} about P_∞ such that $\zeta^{n+1} = \pi$ (hence ζ^{n+1} is globally defined on X although ζ will not be).

Given $(X, \pi, \mathcal{L})$ we define the following collection of functions f_k. For simplicity, assume $\mathcal{L}$ is the divisor line bundle for a positive divisor D, of degree p, whose support does not include P_0 or P_∞. For any integer k there is a unique function $f_k(z,t;P)$ on X with the following properties:

(1) Near P_∞ the function f_k has Fourier series of the form $\zeta^k\exp(z\zeta)(1 + \mathrm{O}(\zeta^{-1}))$.

(2) On $X - \{P_0, P_\infty\}$ it is meromorphic with divisor of poles $\leq D$.
(3) At P_0 the function $\exp(-t\zeta^{-1})f_k$ has an isolated zero of order k.
This definition is very similar to the definition of the two-point scalar Baker function given in [11]. Indeed, using standard arguments from [11] it is easily shown that the $n+1$ functions ψ_j given by (restricting f_j to a neighbourhood of P_∞ and) setting $\psi_j = \zeta^{-j} f_j$ satisfy (14), where $j = 0, \ldots, n$. The corresponding functions q_j and v_j are given by

$$q_j = a_j - a_{j+1}, \qquad v_j = \partial_t a_j,$$

where $\psi_j = \exp(z\zeta)(1 + a_j\zeta^{-1} + \cdots)$.

A comparison between (6) and (14) shows that the function ψ_0 is precisely the scalar Baker function ψ for functions v_j which provide solutions to the complexified Toda lattice (15). To pass to the elliptic Toda lattice we must impose certain reality conditions on X and $\mathcal{L}$ so that each v_j is real-valued and strictly positive along $t = -\bar{z}$. First define $J_R(X)$ to be the identity component of the real subgroup $\{L \in J(X) : \overline{\rho_* L} \simeq L^{-1}\}$ of the Jacobian $J(X)$ of X. The appropriate conditions are:

(4a) X must admit an antiholomorphic involution ρ for which $\overline{\pi(P)}^{-1} = \pi(\rho(P))$, π must have no branch points on the unit circle and ρ must fix every point on X over the unit circle;

(4b) $\mathcal{L} \in \{\mathcal{O}_X(R_+) \otimes L : L \in J_R(X)\}$ where $R_+ + \rho_* R_+$ is the ramification divisor of π on $X - \{P_0, P_\infty\}$ (so R_+ has degree p). This implies that $\mathcal{L}$ is non-special.

For a discussion of these conditions see [18] which corrects an error in [15,17].

Remarks. If (X, π) satisfies these conditions then clearly so does (X, π') where $\pi' = \pi^m$ for any positive integer m. I claim that this is the geometry underlying Miyaoka's iterated solutions [20], in which one observes that if $s_0, \ldots, s_n$ satisfies (1) for $\mathfrak{su}_{n+1}$ then $s_0, \ldots, s_n, \ldots, s_0, \ldots, s_n$ (taken m times) satisfies (1) for $\mathfrak{su}_{m(n+1)}$. Indeed one readily sees that the functions f_k are the same in both cases and $f_{k+j(n+1)} = \pi^j f_k$ for $0 \leq k \leq n$ and $j \in \mathbb{Z}$.

5. Double periodicity

In summary, the aim is to prove the following theorem.

Theorem 5. *Doubly periodic Toda solutions arise by satisfying $2p - 4$ conditions on the $2p - 1$ free parameters available in the choice of (X, P_0, P_∞), whenever $p \geq 2$. To further obtain superconformal 2-tori requires satisfying altogether $2p + 2n - 4$ conditions on the $2p$ free parameters available in (X, π). Whenever (X, π) admits a superconformal 2-torus there will be a $p - 2$ real dimensional moduli space of tori with the same (X, π).*

In fact the situation is slightly different for tori which possess an S^1-symmetry (which is always the case for $p \leq 1$). These arise from the solution of the o.d.e. Toda equations and are discussed at the end of this section.

Let us first examine the periodicity conditions. Suppose we have a pair (q, ψ) arising from the data $(X, \pi, \mathcal{L})$ satisfying conditions (4a) and (4b) and let (s, Φ_ζ) denote the corresponding pair satisfying property (2).

Proposition 2. *Φ_1 is doubly periodic if and only if:* (i) *$q(z)$ is doubly periodic (let the periods be called ξ_1, ξ_2) and* (ii) *at every point $O_0, \ldots, O_n$ for which $\pi(O_k) = 1$ we have $\psi(-r\xi_j; O_k) = 1$ for some positive integer r (which may depend on j).*

Proof. Only (ii) requires explanation. If Φ_1 is doubly periodic then the monodromy matrices $m_j(\zeta)$ satisfy $m_j(1)^r = Id$. Each $m_j(\zeta)$ is ν-equivariant, since Φ_ζ is, which implies that $\det(m_j(\zeta) - \mu \cdot Id)$ is an analytic function of ζ^{n+1} and μ: we will write this as $F_j(\zeta^{n+1}, \mu)$. Now consider the function $F_j(\pi(P), \psi(\xi_j; P)^{-1})$, which is defined and analytic on that part of $X - \{P_0, P_\infty\}$ corresponding to $|\zeta| \geq \epsilon$. By (11) and Proposition 1 this vanishes for $|\zeta| \geq \epsilon^{-1}$, therefore it vanishes identically wherever it is defined. Thus $\psi(\xi_j; P)^{-r} = 1$ at all the points P with $\pi(P) = 1$. Clearly $\psi(\xi; P)^k = \psi(k\xi; P)$ for any integer k (by the monodromy property) which gives us the result.

Conversely, suppose (i) and (ii) both hold, then (i) implies that $\Phi_\zeta^{-1}\partial\Phi_\zeta$ is doubly periodic (by Lemma 2), so Φ_ζ has two monodromy generators $m_j(\zeta)$. When (ii) holds it follows (from the reasoning above) that $m_j(\zeta)^r = Id$ at *every* ζ for which $\zeta^{n+1} = 1$. □

Now I must explain what these conditions mean for the spectral data $(X, \pi, \mathcal{L})$. Let us consider first Proposition 2 (i). Recall the following facts from [15]. The function $q(z, t)$ is actually the restriction of a meromorphic function of infinitely many variables t_k for which $z = t_1$ and $t = t_{-1}$. These variables are coordinates on the (abelian Lie) algebra $\mathcal{G} = C^\omega(E, \mathbb{C})$: any $a \in \mathcal{G}$ has Fourier series $a(\zeta) = \sum t_k\zeta^k$. The homomorphism $\mathcal{G} \to \Lambda_E(\mathfrak{gl}_{n+1}, \nu)$ which maps $a(\zeta)$ to $a(\zeta\Lambda)$ identifies $\mathcal{G}$ with the Lie algebra of Γ (if we work with $G^C = GL_{n+1}$). When q possesses a spectral curve one has a surjective homomorphism $L : \mathcal{G} \to J(X)$ along the kernel of which q is invariant, i.e. q pushes down to a meromorphic function on $J(X)$. This map assigns to $a(\zeta)$ the line bundle $[e^a]$ constructed by taking e^a as a transition function in punctured discs about both P_0 and P_∞ on X. The map L factors through a surjective linear map $l : \mathcal{G} \to H^1(X, \mathcal{O}_X)$.

Now consider the reality conditions required to obtain solutions to (1). The real involution on $\Lambda_C(G^C, \nu)$ corresponds to $a(\zeta) \mapsto \overline{a(\bar{\zeta}^{-1})}$ on $\mathcal{G}$: call the fixed point subspace $\mathcal{G}_R$. The reality conditions (4a) on X are precisely such that the image of $\mathcal{G}_R$ under L is $J_R(X)$. Let $V = \{z\zeta - \bar{z}\zeta^{-1} : z \in \mathbb{C}\} \subset \mathcal{G}_R$, then $q(z, \bar{z})$ is doubly periodic precisely when $L : V \to J_R(X)$ is doubly periodic. When X is smooth of genus $p = 0, 1$ this is always the case (indeed for $p = 0$ we simply get $q = 0$, the vacuum solution). For $p \geq 2$ there are two possibilities: (a) L is doubly periodic but not locally injective, i.e. q is invariant in some direction, or; (b) the image of V under L is a 2-torus. Case (a) occurs precisely when $P_0 + P_\infty$ is a g_2^1, i.e. the divisor of poles of a rational function on X (see the proof below). In that case X is hyperelliptic and we get solutions of the o.d.e. Toda lattice: X is (essentially) a Toda curve in the sense of [19]) and has $p \leq n$ (see the discussion below). The corresponding harmonic tori (if there are any) will be S^1-equivariant. The next result formulates the periodicity conditions in either case. In it $H^1_\rho(X, \mathcal{O})$ denotes the image of $\mathcal{G}_R$ under l, which has dual $H^0_\rho(X, \Omega) = \{\omega\colon \overline{\rho^*\omega} = -\omega\}$.

Proposition 3. *Let X satisfy condition* (4a) *and have arithmetic genus $p \geq 2$. The function $q(z, \bar{z})$ is doubly periodic if and only if either:*

(i) *(X, P_0, P_∞) is a Toda curve and there exists a non-zero $\alpha \in H^1_\rho(X, \mathbb{Z})$ for which*

$$\langle \alpha, \omega\rangle = 0 \quad \text{for all} \quad \omega \in H^0_\rho(X, \Omega(-P_0 - P_\infty)), \tag{16}$$

where $\langle\ ,\ \rangle$ is the Serre duality between $H^1(X, \mathcal{O})$ and $H^0(X, \Omega)$, or;

(ii) *(X, P_0, P_∞) is not a Toda curve and there are two independent solutions α_1, α_2 to* (16). *The first case presents $p - 1$ real equations while the second case presents $2p - 4$ real equations involving only the moduli of X, P_0, P_∞.*

Proof. The duality between $H^1(X, \mathcal{O})$ and $H^0(X, \Omega)$ comes from the bilinear pairing

$$\langle a(\zeta), \omega\rangle = res_{P_0} a\omega + res_{P_\infty} a\omega$$

between $\mathcal{G}$ and $H^0(X, \Omega)$. The subspace $H^0_\rho(X, \Omega(-P_0 - P_\infty))$ denotes all those real holomorphic differentials which have divisor of zeros at least $P_0 + P_\infty$. This is clearly the annihilator of $l(V)$ and has real dimension $p - 1$ if $P_0 + P_\infty$ is a g^1_2, $p - 2$ otherwise. Hence an element α of $H^1_\rho(X, \mathcal{O})$ belongs to $l(V)$ precisely when it satisfies (16). □

Now consider the conditions in Proposition 2(ii). Notice that $\psi(\xi_j; P)$ is unimodular over the unit circle in the ζ^{n+1}-sphere since it provides the eigenvalues for the matrix $m_j(\zeta)$, which is unitary over $|\zeta| = 1$. Therefore the equations (ii) should be counted as 'real' equations for the real curve/covering (X, ρ, π) and the 'real' divisor D. Since the determinant of m_j is 1 everywhere, we deduce that

$$\psi(\xi_j; O_0) \times \cdots \times \psi(\xi_j; O_n) = 1.$$

Therefore these present $2n$ more real equations to be satisfied.

Next we must examine how many parameters we have available to satisfy these conditions. To write down both sets of periodicity conditions in Proposition 2 we require $(X, \pi, \mathcal{L})$. However, we will prove shortly that the periodicity conditions on ψ are independent of the choice of $\mathcal{L}$. Given this result, we only need to count the moduli available in the choice of (X, π).

The pair X, π is one of a finite number characterised by the branch divisor of π (see for example [13]). In the case of general position this amounts to a free choice of $2p$ distinct branch points $\pi(P_1), \ldots, \pi(P_{2p})$ on $\mathbb{CP}^1 - \{0, \infty\}$ (that the number $2p$ follows directly from the Riemann–Hurwitz formula). For X to admit the involution ρ these must be mapped to each other by $\pi \mapsto \bar{\pi}^{-1}$, hence we have at most $2p$ real parameters. If we are only interested in (X, P_0, P_∞) then we are free to scale π by a unimodular constant, hence there are only $2p - 1$ free parameters.

It remains to be shown that the periodicity conditions are independent of the choice of $\mathcal{L}$.

Lemma 3. *Let $\psi_D(z; P)$ denote the scalar Baker function corresponding to the divisor D and let E be any positive divisor on X also of degree p and satisfying the same reality condition. Then for any period ξ of $L(z)$ we have $\psi_E(\xi; P) = \psi_D(\xi; P)$ throughout X.*

Proof. To prove this we must think of ψ_D as a function on $\mathcal{G}_R$ defined as follows. For each $a \in \mathcal{G}_R$ define $\psi_D(a; P)$ to be the unique function on X with the properties: (i) near P_∞ the function $\psi_D(a; P)$ has Fourier series of the form $\exp(a)(1+\mathrm{O}(\zeta^{-1}))$; (ii) on $X - \{P_0, P_\infty\}$ it is meromorphic with divisor of poles $\leq D$; (iii) $\exp(-a)\psi_D(a; P)$ is holomorphic about P_0 and non-zero at P_0. This function has divisor of zeros $D(a)$ where $D(a) - D$ lies in the divisor class of the degree zero line bundle lying beneath the point $l(a)$ in $H^1(X, \mathcal{O})$. Since l is a surjective map every divisor class of degree p contains a positive divisor of the form $D(a)$ for some $a \in \mathcal{G}_R$. In that case $E = D(b)$ for some fixed $b \in \mathcal{G}_R$. Now we observe that, for all $a \in \mathcal{G}_R$

$$\psi_D(a+b) = \psi_D(b)\psi_E(a). \tag{17}$$

The proof is straightforward. The right-hand side clearly has the same exponential behaviour about P_0 and P_∞ as the left-hand side. Moreover, $\psi_D(b)$ has divisor of zeros E (since the reality conditions mean that E is non-special and is therefore the unique positive divisor in its divisor class) hence both sides have divisor of poles D on $X - \{P_0, P_\infty\}$. The result now follows from the uniqueness of the Baker function. In particular this means, for $j = 1, 2$,

$$\psi_D(a+\xi_j) = \psi_D(\xi_j)\psi_D(a) \tag{18}$$

for all $a \in \mathcal{G}_R$ since $D(\xi_j) = D$. Combining (17) with (18) results in

$$\psi_E(a - b + \xi_j) = \psi_D(\xi_j)\psi_E(a-b).$$

Setting $a = b$ in this gives us our result, since $\psi_E(0) = 1$. $\square$

As a corollary we deduce the final part of Theorem 5, since every superconformal 2-torus is determined by a collection of data (X, π, D). By Lemma 3 every point on $\{\mathcal{O}_X(D) \otimes L : L \in J_R(X)\}$ must give rise to a superconformal 2-torus if $\mathcal{O}_X(D)$ does. But $J_R(X)$ also contains the 2-parameter subgroup $L(V)$ corresponding to conformal automorphisms of the 2-torus and we take the moduli space to be the quotient of $J_R(X)$ by this subgroup.

Remarks. Elsewhere in [18] I have shown that we can combine the two types of periodicity condition by introducing the singularisation $X_\mathfrak{o}$ of X obtained by identifying together all the points of the divisor $\mathfrak{o} = O_0 + \cdots + O_n$ of zeros of $\pi - 1$. I claim the periodicity conditions of Proposition 2 are all satisfied precisely when there are two independent solutions $\alpha_1, \alpha_2 \in H^1_\rho(X_\mathfrak{o}, \mathbb{Z})$ to (16) with Ω replaced by $\Omega(\mathfrak{o})$ – the sheaf of regular differentials on $X_\mathfrak{o}$ (see [22, Chap. IV, Section 3]). This can be seen directly here in the following way. Set

$f = \psi(\xi_j; P)$ which, by the proof of Lemma 3, is holomorphic and non-vanishing on $X - \{P_0, P_\infty\}$. It follows that $\mathrm{d}ff^{-1}$ is a differential of the second kind with poles only at P_0, P_∞. An equivalent way of expressing Proposition 2(ii) is that

$$\frac{1}{2\pi \mathrm{i}} \int_{O_0}^{O_k} \mathrm{d}ff^{-1}$$

is rational for each k. By reciprocity this amounts to rationality conditions on the values at P_0 and P_∞ of a vector of differentials spanning $H^0(\Omega(\mathfrak{o}))$.

Toda Curves. We will count the number of parameters available to find tori for generic Toda curves with $p \geq 2$. Let (X, P_0, P_∞) be a Toda curve, then since ρ preserves $P_0 + P_\infty$ there is a rational function μ with that pole divisor and satisfying $\rho_*\mu = \bar{\mu}$: it is unique up to real affine transformations. The (unique) hyperelliptic involution σ determined by $P_0 + P_\infty$ swaps sheets of $\mu : X \to \mathbb{CP}^1$ and therefore σ commutes with ρ. Clearly $\pi \cdot \sigma_*\pi$ and $\pi + \sigma_*\pi$ are functions of μ, indeed the former must be a unimodular constant. By rescaling we obtain $\sigma_*\pi = \pi^{-1}$ and $\pi + \sigma_*\pi = b(\mu)$ where $b(\bar{\mu}) = \overline{b(\mu)}$. Hence we have a hyperelliptic curve with affine equation

$$\pi - b(\mu) + \pi^{-1} = 0.$$

This curve is birationally equivalent to (and no smoother than) X. Now $b(\mu) = 0$ when $\pi^2 = -1$. Since both i and $-i$ are not branch points of π (by condition (4a)) there are $2n + 2$ distinct points on X where $b(\mu) = 0$, and these come in pairs $S_k, \sigma(S_k)$. Therefore $b(\mu)$ has degree $n + 1$ whence the curve above has arithmetic genus n (so $p \leq n$). Also, by condition (4a), every S_k is ρ-fixed so that $b(\mu)$ has $n + 1$ distinct real roots. Since we cannot assume $b(\mu)$ is monic it depends on $n + 2$ parameters. However, we can scale one away if we only want the number of parameters for (X, P_0, P_∞). Therefore in the generic case, $p = n$, we have $n + 1$ real parameters. For (X, π) we must include the two scaling parameters (a unimodular scaling for π and a real scaling for $b(\mu)$) whence we have $n + 3$ real parameters.

To count the periodicity conditions it is best to use the remark above. The Toda solution has one trivial period corresponding to $\alpha_1 \in H^1(X_{\mathfrak{o}}, \mathbb{Z})$ satisfying $\langle \alpha_1, \omega \rangle = 0$ for every regular differential on X. Therefore when we replace Ω by $\Omega(\mathfrak{o})$ in (16) α_1 satisfies $p - 1$ of those equations automatically. Since $\dim H^1(X, \Omega(\mathfrak{o} - P_0 - P_\infty)) = p + n - 2$ (for $p > 1$) we have at most $n - 1$ independent conditions on α_1. The non-trivial period in the Toda solution will correspond to α_2 satisfying the full $p + n - 2$ equations, so the total number of equations between them is $3n - 3$ when $p = n$. Comparing this with the number of degrees of freedom $n + 3$ we seem to have a non-negative net parameter count for $n \leq 3$, which is only a slight improvement on the general case, where $n \leq 2$.

Example.

(i) When $p = 0$ the curve X is the Riemann sphere with rational parameter ζ and $\pi = \zeta^{n+1}$. The corresponding Toda solution is the vacuum solution: $q = 0$. The scalar Baker function is simply $\psi(z, \bar{z}; \zeta) = \exp(z\zeta - \bar{z}\zeta^{-1})$. It is not hard to show this can have two independent periods in z at every point over $\zeta^{n+1} = 1$ if and only if $n = 1, 2, 3, 5$. The corresponding 2-tori are the flat tori described in [5].

(ii) Next consider $p = 1$ and $n = 1$. Then $\Omega(D - P_0 - P_\infty)$ has no global sections since $D \sim 2P_0 \not\sim P_0 + P_\infty$. Moreover, it is easy to see that $H^1_p(X_p, \mathbb{Z}) \simeq \mathbb{Z}^2$, therefore the periodicity conditions are non-trivially satisfied. The harmonic tori are of course the Gauss maps of Delauney surfaces in $\mathbb{R}^3$.

6. Toda fields which possess a spectral curve

In this section we will see that every Toda solution of finite type possesses a spectral curve of the type required. It follows that every superconformal 2-torus comes from the construction above. First it is shown that every solution of finite type has a finite-dimensional Γ-orbit (in the set-up of Theorem 4) and then it is shown that every finite-dimensional Γ-orbit in $\mathcal{M}/K$ is identifiable with the Jacobian of a spectral curve.

Following [4] we observe that on the finite-dimensional manifold Λ_m (recall Section 2) we can define two real o.d.e.'s

$$\partial\eta = [\eta, \mathrm{i}(\zeta\eta_m + \eta_{m-1})], \qquad \bar{\partial}\eta = [\eta, -\mathrm{i}(\zeta^{-1}\eta_{-m} + \eta_{1-m})]. \tag{19}$$

where each $(\bar{\eta}, \eta)$ in Λ_m is identified with the Laurent polynomial η. These o.d.e.'s are in fact Hamiltonian and their flows commute. Together the pair has a finite-dimensional space of solutions $(\bar{\eta}, \eta) : \mathbb{R}^2 \to \Lambda_m$, each determined uniquely by its initial conditions. Given a doubly periodic solution matrix s for the elliptic Toda equations Theorem 3 says there is a solution η to (19) satisfying $\mathrm{i}\eta_m = -s^{-1}\Lambda s$ and $\mathrm{i}\eta_{m-1} = -\partial \ln s$. In particular η determines s uniquely up to a scaling (indeed, up to a scaling by some power of ω, in view of Theorem 4). Hence only a finite-dimensional family of doubly periodic solutions of the Toda equations may correspond to each $m \equiv 1 \bmod (n+1)$. Now we prove:

Lemma 4. *Let $s(z, \bar{z})$ be a doubly periodic solution to the elliptic Toda equations and let $\tilde{s}$ be the point in $\mathcal{M}/K_0$ it comes from according to Theorem 4. Then $\tilde{s}$ has a finite-dimensional Γ-orbit in $\mathcal{M}/K_0$.*

Proof. It is enough to show that the Γ_R-orbit of $\tilde{s}$ is finite-dimensional, where Γ_R is real subgroup of Γ with respect to the real involution on $\Lambda_C(G^C, \nu)$. By Theorem 4 the Γ_R-orbit of $\tilde{s}$ corresponds to a smooth family $s(z, \bar{z}, \mathbf{t})$ of solutions to the elliptic Toda equations, parameterised by real variables corresponding to the higher flows in the Toda hierarchy. Since all these flows commute (Γ_R is abelian) every solution in this family is also doubly periodic. But according to the discussion above each solution in this family corresponds to a solution of the o.d.e's (19) above, for some fixed $m \equiv 1 \bmod (n+1)$. Since the space of these solutions is finite-dimensional so is the family $s(z, \bar{z}, \mathbf{t})$. It follows that the Γ_R-orbit of $\tilde{s}$ must also be finite-dimensional. □

Now we will prove the principal result about solutions to the complexified Toda equations.

Theorem 6. *Every finite-dimensional Γ-orbit in $\mathcal{M}/K$ is identifiable with the Jacobian of a complete algebraic curve of the type described earlier.*

One then knows from [15,16] that this curve is precisely the spectral curve in the sense used here. To prove this it is easiest to make use of the note after Theorem 4 and work with $G^{\mathbb{C}} = GL_{n+1}$ throughout. The idea is to identify $\mathcal{M}$ with an infinite-dimensional Grassmanian analogous to that used by Segal and Wilson [21]. Define

$$H_\sigma = \{f \in L^2(C, \mathbb{C}^{n+1}):\ f(\omega\zeta) = f(\zeta)\sigma^{-1}\},$$

where we think of elements of $\mathbb{C}^{n+1}$ as row vectors. The group $\Lambda_C(G^{\mathbb{C}}, \nu)$ acts on this space by $g \circ f = fg^{-1}$. Define H_E to be the subspace of H_σ consisting of boundaries of holomorphic functions $f : E \to \mathbb{C}^{n+1}$ and define

$$Gr = \{g^{-1} \circ H_E:\ g \in \Lambda_C(G^{\mathbb{C}}, \nu)\}$$

then we clearly have $Gr \simeq \mathcal{M}$. Now let $\mathcal{A}$ be the (abelian Lie) algebra of real-analyiic maps $C \to \mathbb{C}$. This acts on H_σ through its isomorphism with the centraliser of $\zeta\Lambda$ in $\Lambda_C(\mathfrak{g}^{\mathbb{C}}, \nu)$ (this is just the extension of the isomorphism of $\mathcal{G} \subset \mathcal{A}$ with the Lie algebra of Γ). Specifically, for $a(\zeta) \in \mathcal{A}$ and $f \in H_\sigma$ we define $a \circ f = -f \cdot a(\zeta\Lambda)$: this makes H_σ into an $\mathcal{A}$-module.

The reason for introducing $\mathcal{A}$ is that to each $W \in Gr$ we can associate the subalgebra

$$\mathcal{A}_W = \{a \in \mathcal{A}: a \circ W \subset W\}.$$

Ultimately we will show that this is (essentially) the coordinate ring of the affine curve $X - \{P_0, P_\infty\}$ whenever the corresponding point in $\mathcal{M}/K$ has finite-dimensional Γ-orbit.

Let $\mathcal{A}_I$ denote the subalgebra of boundaries of functions holomorphic on I and vanishing at $\zeta = \infty$. It is not hard to see that $K = \exp(\mathcal{A}_I)$ and $\mathcal{A} = \mathcal{A}_I \oplus \mathcal{G}$. Therefore Γ-orbits in Gr/K are identical to $\exp(\mathcal{A})$-orbits.

Lemma 5. *For $W \in Gr$ let $[W]$ denote its K-orbit as a point of Gr/K and let $\mathcal{I}_{[W]}$ denote the isotropy group of $[W]$ for the action of* $\exp(\mathcal{A})$ *on Gr/K. Then the Lie algebra of $\mathcal{I}_{[W]}$ is $\mathcal{A}_I \oplus \mathcal{A}_W$.*

Proof. An element a of $\mathcal{A}$ belongs to the Lie algebra of $\mathcal{I}_{[W]}$ precisely when there exists, for each $t \in \mathbb{C}$, some $b_t \in \mathcal{A}_I$ depending smoothly on t such that $\exp(ta - b_t)W = W$. Taking derivatives at $t = 0$ shows that $a \in \mathcal{A}_I + \mathcal{A}_W$. Moreover, it is clear from this argument that $\mathcal{A}_I + \mathcal{A}_W$ is contained in the Lie algebra of $\mathcal{I}_{[W]}$. Now we recall from [15] that K acts freely on $\mathcal{M}$ so that $\mathcal{A}_I \cap \mathcal{A}_W = \{0\}$. □

As a consequence we have the exact sequence of abelian Lie algebras

$$\begin{array}{ccccccc} 0 \to & \mathcal{A}_I \oplus \mathcal{A}_W & \to & \mathcal{A} & \to \mathcal{H}_{[W]} \to 0 \\ & (b, a) & \mapsto & a - b & \end{array} \tag{20}$$

The heart of the proof of Theorem 6 is to show that $\mathcal{H}_{[W]}$ is finite-dimensional precisely when this sequence equates $\mathcal{H}_{[W]}$ with $H^1(X, \mathcal{O})$ for the completion of the affine curve 'Spec($\mathcal{A}_W$)'. This result can be found in [17] but it will help to summarise the proof here.

To make proper sense of Spec($\mathcal{A}_W$) we follow [21] and work with the subalgebra $\mathcal{A}^{\mathrm{alg}}$ of functions of finite order. An element $(a_1, a_2) \in \mathcal{A}$ has *finite order* if $a_1(\zeta)$ extends meromorphically to a neighbourhood of $\zeta = 0$ *and* a_2 extends meromorphically to a neighbourhood of $\zeta^{-1} = 0$. Likewise, we use W^{alg} to denote the finite-order elements of $W \subset H$. Finally, we let $\mathcal{C} \subset \mathcal{A}$ denote the subalgebra of functions of ζ^{n+1} only. It is readily shown that W^{alg} is a torsion free $\mathcal{A}_W^{\mathrm{alg}}$-module from which we deduce that $\mathcal{A}_W^{\mathrm{alg}}$ is a finitely generated (free) $\mathcal{C}^{\mathrm{alg}}$-module with rank $m \le n+1$ and, moreover, it is an integral domain.

It follows that there is a finite, surjective morphism $\pi : \mathrm{Spec}(\mathcal{A}_W^{\mathrm{alg}}) \to \mathrm{Spec}(\mathcal{C}^{\mathrm{alg}})$. Since $\mathcal{C}^{\mathrm{alg}} \simeq \mathbb{C}[\zeta^{n+1}, \zeta^{-n-1}]$ it may be thought of as the coordinate ring of the ζ^{n+1}-sphere punctured at 0 and ∞, hence we have a finite degree covering of this punctured sphere. Clearly the degree of this covering can be at most $n+1$. By the remarks above Spec($\mathcal{A}_W^{\mathrm{alg}}$) is irreducible and may be completed by adding non-singular points P_0 and P_∞ over 0 and ∞, respectively. We call the completion X. These added points are determined by the valuations on the fraction field of $\mathcal{A}_W^{\mathrm{alg}}$ corresponding to poles of a_1 and a_2, respectively, for $(a_1, a_2) \in \mathcal{A}_W^{\mathrm{alg}}$. Using these valuations one shows that the vector space $\mathcal{H}_{[W]}$ is finite-dimensional if and only if the $\mathcal{C}^{\mathrm{alg}}$-module A_W^{alg} has rank $n+1$. In that case the covering $\pi : X \to \mathbb{P}$ behaves like $\zeta \mapsto \zeta^{n+1}$ about P_0 so that $\mathcal{A}_I$ corresponds to the algebra of holomorphic functions in a neighbourhood of P_0 and P_∞ (vanishing at P_∞). It follows that the sequence (20) computes $H^1(X, \mathcal{O})$.

Now Theorem 6 follows from the observation that $[W]$ uniquely determines (and is determined by) the data $(X, \pi, \mathcal{L})$ where $\mathcal{L}$ is a line bundle over X (or, if X is singular, perhaps merely a rank 1 torsion free coherent sheaf). The bigraded $\mathcal{A}_W^{\mathrm{alg}}$-module W^{alg} determines $\mathcal{L}$ equipped with trivialisations over P_0 and P_∞: passing to $[W]$ corresponds to discarding these trivialisations. From [15] we know that the correspondence between these two identifies the exp($\mathcal{A}$)-action with the $J(X)$-action on the Picard variety of X (or, more generally, the moduli space of rank 1 torsion free coherent sheaves over X). If $\mathcal{L}$ is a line bundle we know immediately that its $J(X)$-orbit is identifiable with $J(X)$ itself. More generally, the $J(X)$-orbit is identifiable with $J(X')$, where X' is the least singular curve from which $\mathcal{L}$ could come by direct image. However, the dimension of $J(X')$ must equal $\dim \mathcal{H}_{[W]} = \dim J(X)$, hence $X \simeq X'$.

Acknowledgements

I am grateful to J. Bolton, F.E. Burstall, F. Pedit and L.M. Woodward for keeping me informed of their work [4–6,8,9] and for their many enlightening remarks. This article was completed in January 1994 with the support of SERC research grant GR/J33043.

Appendix A. Spectral curves and Killing fields

This appendix explains the connection between the spectral curve used here and alternative definition given in [12]. There one finds a concrete construction for an algebraic curve given a harmonic 2-torus in a symmetric space, and it is clear how this adapts to our case. As in [7], the emphasis is placed on *polynomial Killing fields*. So my first task is to explain how these fit into the picture here.

Let us fix a superconformal 2-torus in $\mathbb{CP}^n$ with extended Toda frame Φ, and set $\alpha = \Phi^{-1}\,\mathrm{d}\Phi$ (we will drop the subscript ζ in this discussion). By Theorem 3 we may always choose Φ to be given by a factorisation $g\Phi^{(0)} = \Phi\chi$ for some $g \in \Lambda(G, \nu)$ (where χ takes values in $\Lambda_{I,D}$). We will say an element of $\Lambda_C(\mathfrak{g}^{\mathbb{C}}, \nu)$ has *finite order* if its component in $\Lambda_E(\mathbf{g}^{\mathbb{C}}, \nu)$ (given the usual splitting) is a Laurent polynomial in ζ. Throughout this section we work with $\mathfrak{g}^{\mathbb{C}} = \mathfrak{gl}_{n+1}$. A *formal Killing field* for α is a loop algebra-valued analytic function $\eta(z, \bar{z})$ which satisfies

$$\mathrm{d}\eta = [\eta, \alpha] \tag{A.1}$$

and is of (bounded) finite order for every $z, \bar{z}$. Notice that η is not restricted to be real here (i.e. with respect to the antiholomorphic involution on $\Lambda_C(\mathfrak{g}^{\mathbb{C}}, \nu)$). We will say η is a *polynomial* Killing field whenever it takes values in $\Lambda_E(\mathfrak{g}^{\mathbb{C}}, \nu)$ (which is precisely the case for which it is a Laurent polynomial, by our definition). One of the principal results of [7], when adapted to this context (as in [4]), shows that the existence of a formal Killing field implies the existence of real polynomial Killing fields satisfying the extra condition in (5) which guarantees that Φ is of finite type.

It is elementary to show that the formal Killing fields form a Lie algebra. In fact they form a commutative Lie algebra isomorphic to $\mathcal{A}^{\mathrm{alg}}$, the subalgebra of finite-order elements in $\mathcal{A}$. Here we view $\mathcal{A}$ as the centraliser in $\Lambda_C(\mathfrak{g}^{\mathbb{C}}, \nu)$ of $\zeta\Lambda$. Under this isomorphism, the polynomial Killing fields correspond to $\mathcal{A}^{\mathrm{alg}}_W$, where W now denotes the coset $\Lambda_E(G^{\mathbb{C}}, \nu)\cdot g$. This isomorphism is given by

$$\begin{aligned} Ad\chi : \mathcal{A}^{\mathrm{alg}} &\to \{\text{formal Killing fields}\} \\ a &\mapsto \eta = Ad\chi \cdot a. \end{aligned} \tag{A.2}$$

In order to prove these statements it is useful to have the following lemma. Let $\alpha^{(0)} = \Phi^{(0)-1}\,\mathrm{d}\Phi^{(0)}$.

Lemma A.1. *Let $a(z, \bar{z})$ be a formal Killing field for $\alpha^{(0)}$. Then $a \in \mathcal{A}^{\mathrm{alg}}$ (in particular $\mathrm{d}a = 0$).*

Proof. Write $a = (a_1, a_2)$. We will show that $\mathrm{d}a_1 = 0$ and a_1 commutes with Λ. A similar procedure will also work for a_2, proving the lemma.

Let us write $a_1 = \sum_{-\infty}^{k} a_{1j}\zeta^{-j}$. From the Killing field equation we see that

$$\partial a_{1k} = 0, \quad \partial a_{1j} = [\Lambda, a_{1j-1}], \quad \bar{\partial} a_{1j+1} = [a_{1j}, \Lambda^*], \quad [a_{1k}, \Lambda^*] = 0, \tag{A.3}$$

where $j < k$. The proof requires the observation that the image and kernel of $Ad\Lambda$ equal those of $Ad\Lambda^*$ and these intersect only on $\{0\}$. The latter follows from the fact that Λ is semisimple. Using this observation it follows that $\bar{\partial}a_{1k} = 0$, and therefore a_{1k} is constant and commutes with Λ. This means that the equations above are also true with k replaced by $k-1$. By induction we see that for every j, a_{1j} is constant and commutes with Λ. □

Now let us prove:

Proposition A.1. *The map* (A.2) *is an isomorphism of Lie algebras which identifies* $\mathcal{A}_W^{\mathrm{alg}}$ *with the subalgebra of polynomial Killing fields for* α.

Proof. Using $\chi = \Phi^{-1}g\Phi^{(0)}$ we see that, for each $a \in \mathcal{A}$, $Ad\chi \cdot a = Ad\Phi^{-1} \cdot \eta_0$ where $\eta_0 = Adg \cdot a$ is independent of $z, \bar{z}$. Therefore

$$\mathrm{d}(Ad\chi \cdot a) = Ad\Phi^{-1} \cdot ([\Phi\,\mathrm{d}\Phi^{-1}, \eta_0] + \mathrm{d}\eta_0) = [Ad\chi \cdot a, \alpha]. \tag{A.4}$$

Moreover, $Ad\chi \cdot a$ has finite order whenever a does, since χ extends holomorphically into I. Conversely, given a formal Killing field η one easily shows that $\mathrm{d}(Ad\Phi \cdot \eta) = 0$ and therefore $\mathrm{d}(Ad\Phi^{(0)} \cdot a) = 0$ for $a = Ad\chi^{-1} \cdot \eta$. When this is expanded we see that a is a formal Killing field for $\alpha^{(0)}$. By Lemma A.1 this implies $a \in \mathcal{A}^{\mathrm{alg}}$.

Now let us see that η takes values in $\Lambda_E(\mathfrak{g}^{\mathbb{C}}, \nu)$ if and only if $a \in \mathcal{A}_W^{\mathrm{alg}}$. We know $a \in \mathcal{A}_W$ precisely when $We^{ta} = W$ for all $t \in \mathbb{C}$. But since e^{ta} commutes with $\Phi^{(0)}$ this is equivalent to $\Lambda_E(G^{\mathbb{C}}, \nu)\chi e^{ta} = \Lambda_E(G^{\mathbb{C}}, \nu)\chi$. Therefore $a \in \mathcal{A}_W$ if and only if $\exp(tAd\chi \cdot a)$ takes values in $\Lambda_E(G^{\mathbb{C}}, \nu)$ for all t, which proves the result. □

Notice that since χ is real a Killing field η is real if and only if a is.

It is now possible to refer to the commutative algebra of polynomial Killing fields, where the ring product is ordinary matrix multiplication. However, since each Killing field is a function of $z, \bar{z}$ it is more convenient to look at this as a parameterised family $R(z, \bar{z})$ of commutative algebras, each isomorphic to $\mathcal{A}_W^{\mathrm{alg}}$. In particular, we will use R to denote $R(0, 0)$: the algebra of polynomial Killing fields evaluated at $z, \bar{z} = 0$.

The spectral curve in [12] is defined as follows. Set

$$Y = \{(\zeta, [v]) \in \mathbb{C}^* \times \mathbb{CP}^n :\ \eta_0(\zeta)v = \mu v,\ \forall \eta_0 \in R,\ \exists \mu \in \mathbb{C}\}.$$

The spectral curve is then defined to be the compactification $\tilde{Y}$ of Y in $\mathbb{CP}^1 \times \mathbb{CP}^n$. Since elements of R are simultaneously diagonalisable we see that Y is independent of the choice of η_0. However, at first sight it is not clear that this map is well-defined throughout Y_η: problems might arise at points where the eigenspaces coincide. In [12] attention was restricted to the generic case, where eigenspaces coincide at worst in pairs, i.e. where the ramification points of Y over $\mathbb{CP}^1$ (under projection on the first factor) are all simple.

Now let us see the connection between X and $\tilde{Y}$. The ν-equivariance of η_0 induces a fixed-point free automorphism (which we shall also call ν) on Y inherited from the automorphism $(\zeta, [v]) \mapsto (\omega\zeta, [\sigma v])$ of $\mathbb{C}^* \times \mathbb{CP}^n$. It can be shown quite easily that R is the coordinate ring for Y/ν, since R is ring of all ν-equivariant regular endomorphisms of the 'eigenspace line bundle' over Y (this is the pull-back of the tautological line bundle over $\mathbb{CP}^n$). Therefore $X_A \simeq Y/\nu$ and in fact $X \simeq \tilde{Y}/\nu$ so that $\tilde{Y}$ is an unramified cover of X. Indeed it is this curve which Hitchin uses in [14]. This explains why he only finds harmonic maps into $\mathbb{CP}^1$ for curves of odd genus: the genus of an unramified double cover of X is $2p-1$. One knows (from e.g. [22]) that the subgroup $J(\tilde{Y})_\nu$ of ν-fixed points in $J(\tilde{Y})$ is a finite cover of $J(X)$ with covering group $\mathbb{Z}_{n+1}$. It represents the ν-equivariant line bundles of degree zero over $\tilde{Y}$. It was shown in [15] that, whereas $J(X)$ is identifiable with a finite-dimensional orbit in $\mathcal{M}/K$, this group $J(\tilde{Y})_\nu$ is identifiable with the corresponding orbit in $\mathcal{M}/K_0$, which is the dressing space of Toda solutions.

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